Justification of the coupled-mode approximation for a nonlinear elliptic problem with a periodic potential

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Abstract

Coupled-mode systems are used in physical literature to simplify the nonlinear Maxwell and Gross-Pitaevskii equations with a small periodic potential and to approximate localized solutions called gap solitons by analytical expressions involving hyperbolic functions. We justify the use of the one-dimensional stationary coupled-mode system for a relevant elliptic problem by employing the method of Lyapunov–Schmidt reductions in Fourier space. In particular, existence of periodic/anti-periodic and decaying solutions is proved and the error terms are controlled in suitable norms. The use of multi-dimensional stationary coupled-mode systems is justified for analysis of bifurcations of periodic/anti-periodic solutions in a small multi-dimensional periodic potential.

1 Introduction

Gap solitons are localized stationary solutions of nonlinear elliptic problems existing in the spectral gaps of the Schrödinger operator associated with a periodic potential. In particular, gap solitons have been considered in two problems of modern mathematical physics, the complex-valued Maxwell equation

$$\nabla^2 E - \frac{n^2(x,|E|^2)}{c^2} E_{tt} = 0 \tag{1.1}$$

and the Gross-Pitaevskii equation

$$iE_t = -\nabla^2 E + V(x)E + \sigma |E|^2 E, \tag{1.2}$$

where $E(x,t) : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{C}$ and $\nabla^2 = \partial_{x_1}^2 + ... + \partial_{x_N}^2$. For applications of the complex-valued Maxwell equation (1.1), E is a complex amplitude of the electric field vector, c is the speed of light, and $n(x,|E|^2)$ is the refractive index. The scalar equation (1.1) is valid in the space of one and two dimensions ($N = 1, 2$) for so-called TE modes but not in the space of three dimensions ($N = 3$), where the system of Maxwell equations for a vector-valued function E must be used [17]. For applications of the Gross–Pitaevskii equation (1.2), E is the mean-field amplitude, σ is the scattering length, and

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$V(x)$ is the trapping potential. The scalar equation (1.2) is the mean-field model valid in the space of three dimensions ($N = 3$) and it can be used in the space of one and two dimensions ($N = 1, 2$) under additional assumptions [16].

Stationary solutions of the Maxwell and GP equations are found from the elliptic problem

$$\nabla^2 U + \omega^2 U + \epsilon W(x) U = \sigma |U|^2 U, \tag{1.3}$$

where $U(x) : \mathbb{R}^N \mapsto \mathbb{C}$ and $(\omega^2, \epsilon, \sigma)$ are parameters. The elliptic problem (1.3) is related to the GP equation (1.2) by an exact reduction $E = U(x) e^{-i\omega^2 t}$, where the potential $V(x)$ is represented by $V = -\epsilon W(x)$. The same problem is related to the Maxwell equation (1.1) with the refractive index $n(x, |E|^2) = n_0^2 (1 + \mu W(x) + \nu |E|^2)$ by the exact reduction $E = U(x) e^{-ic\omega t / n_0}$, where parameters are related by $\epsilon = \omega^2 \mu$ and $\sigma = -\omega^2 \nu$.

Let us consider the elliptic problem (1.3) with a real-valued bounded potential $W(x)$, which is periodic in each variable x_j, $\forall j$. The associated Schrödinger operator $L = -\nabla^2 - \epsilon W(x)$ is defined on $C_0^\infty(\mathbb{R}^N)$ and is extended to a self-adjoint operator which maps continuously $H^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$. Therefore, the spectrum $\sigma(L)$ is real. Suppose that the absolutely continuous part of the spectrum of L has a gap of finite size on the real spectral axis. For parameters inside the spectral gap, localized solutions of the elliptic problem (1.3) were proved to exist in the relevant variational problem [19]. (Earlier works on bifurcations of gap solitons can be found in [2, 10, 18].) According to Theorem 1.1 of [19], there exists a weak solution $U(x)$ in $H^1(\mathbb{R}^N)$, which is (i) real-valued, (ii) continuous on $x \in \mathbb{R}^N$ and (iii) decays exponentially as $|x| \to \infty$.

We investigate more precise information on properties of the gap soliton $U(x)$ by working with the asymptotic limit of small ϵ. When $\epsilon = 0$, the purely continuous spectrum of $L_0 = -\nabla^2$ is non-negative and no finite gaps of $\sigma(L_0)$ exist. However, when $N = 1$ and $0 \neq \epsilon \ll 1$, narrow gaps of $\sigma(L)$ diverge from a sequence of resonant points on a real axis and gap solitons may bifurcate inside these narrow gaps. The coupled-mode system has been used in the physical literature since the 1980s to characterize this symmetry-breaking bifurcation of the spectrum $\sigma(L)$ and to approximate one-dimensional gap solitons of the problem (1.3) with $N = 1$ [22]. Therefore, our work deals mainly with the case $N = 1$. We consider the potential function $W(x)$ according to the following assumption.

Assumption 1 *Let $W(x)$ be a smooth 2π-periodic function with zero mean and symmetry $W(-x) = W(x)$ on $x \in \mathbb{R}$. The Fourier series representation of $W(x)$ is*

$$W(x) = \sum_{m \in \mathbb{Z}} w_{2m} e^{imx}, \quad \textit{such that} \quad \sum_{m \in \mathbb{Z}} (1 + m^2)^s |w_{2m}|^2 < \infty, \ \ \forall s > \frac{1}{2}, \tag{1.4}$$

where $w_0 = 0$ and $w_{2m} = w_{-2m} = \bar{w}_{2m}$, $\forall m \in \mathbb{N}$.

It will be clear from Proposition 2 that the sequence of resonant points of $\sigma(L)$ for $L = -\partial_x^2 - \epsilon W(x)$ is located at $\omega = \omega_n = \frac{n}{2}$, $n \in \mathbb{Z}$, so that a small gap in the spectrum $\sigma(L)$ bifurcates generally from each point $\omega = \omega_n$, $n \in \mathbb{N}$ and a semi-infinite gap exists near $\omega = \omega_0 = 0$. A formal asymptotic solution of the elliptic problem (1.3) near a resonant point $\omega = \omega_n$, $n \in \mathbb{N}$, is given by

$$U(x) = \sqrt{\epsilon} \left[a(\epsilon x) e^{\frac{inx}{2}} + b(\epsilon x) e^{-\frac{inx}{2}} + \mathrm{O}(\epsilon) \right], \qquad \omega^2 = \frac{n^2}{4} + \epsilon \Omega + \mathrm{O}(\epsilon^2), \tag{1.5}$$

where the vector $(a, b) : \mathbb{R} \mapsto \mathbb{C}^2$ satisfies the coupled-mode system with parameter $\Omega \in \mathbb{R}$

$$\left\{ \begin{array}{l} ina' + \Omega a + w_{2n} b = \sigma(|a|^2 + 2|b|^2)a, \\ -inb' + \Omega b + w_{2n} a = \sigma(2|a|^2 + |b|^2)b, \end{array} \right. \tag{1.6}$$

and the derivatives are taken with respect to $y = \epsilon x$.

The coupled-mode system (1.6) can be used for analysis of bifurcations of periodic and anti-periodic solutions near narrow gaps in the spectrum $\sigma(L)$. When the solutions (a, b) of system (1.6) are y-independent and the representation (1.5) is used, the solution $\phi(x)$ of the elliptic system (1.3) at the leading order is periodic in x for even $n \in \mathbb{N}$ and anti-periodic for odd $n \in \mathbb{N}$. It follows from system (1.6) that a general family of y-independent small solutions (a, b) near the point $(0, 0)$ has the form $a = ce^{i\theta}$ and $b = \pm ce^{i\theta}$, where $c \in \mathbb{R}$, $\theta \in \mathbb{R}$, and the nonlinear dispersion relation holds in the form

$$\Omega \pm w_{2n} - 3\sigma c^2 = 0. \tag{1.7}$$

The dispersion relation (1.7) with $c = 0$ shows that the eigenvalues for periodic/anti-periodic solutions in the linear spectrum of the operator $L = -\partial_x^2 + \epsilon W(x)$ diverge from the point $\omega^2 = \frac{n^2}{4}$ and result in a narrow gap in the spectrum $\sigma(L)$ which lies in the interval

$$\frac{n^2}{4} - |\epsilon w_{2n}| < \omega^2 < \frac{n^2}{4} + \epsilon|\epsilon w_{2n}|. \tag{1.8}$$

The nonlinear dispersion relation (1.7) with $c \neq 0$ shows that the nonlinear periodic/anti-periodic solutions bifurcate to the right of the boundaries $\Omega = \pm w_{2n}$ for $\sigma = +1$ and to the left for $\sigma = -1$. We justify the persistence of the leading-order results (1.7) and (1.8) by using the method of Lyapunov–Schmidt reductions in the discrete weighted space $l_s^2(\mathbb{Z}')$ equipped with the norm

$$\mathbf{U} \in l_s^2(\mathbb{Z}') : \quad \|\mathbf{U}\|^2_{l_s^2(\mathbb{Z}')} = \sum_{m \in \mathbb{Z}'} \left(1 + \frac{m^2}{4}\right)^s |U_m|^2 < \infty. \tag{1.9}$$

Here $\mathbb{Z}'$ is a set of either even or odd numbers and $\mathbf{U}$ is a set of Fourier coefficients $\{U_m\}_{m \in \mathbb{Z}'}$ in the Fourier series

$$U(x) = \sqrt{\epsilon} \sum_{m \in \mathbb{Z}'} U_m e^{\frac{i}{2} mx}, \qquad U_m = \frac{1}{2\pi\sqrt{\epsilon}} \int_0^{2\pi} U(x) e^{-\frac{i}{2} mx} dx, \tag{1.10}$$

where the factor $\sqrt{\epsilon}$ is introduced for convenience. The function $U(x)$ is periodic if the set $\mathbb{Z}'$ is even and it is anti-periodic if $\mathbb{Z}'$ is odd. By the Sobolev inequality, if $\mathbf{U} \in l_s^2(\mathbb{Z}')$ with $s > \frac{1}{2}$, then the series (1.10) converges absolutely and uniformly in the space of bounded continuous functions $C_b^0(\mathbb{R})$ according to the bound

$$\sum_{m \in \mathbb{Z}'} |U_m| \leq \sum_{m \in \mathbb{Z}'} (1 + m^2)^s |U_m|^2 + \sum_{m \in \mathbb{Z}'} \frac{1}{4(1 + m^2)^s} < \infty, \tag{1.11}$$

where we have used the inequality $|a||b| \leq |a|^2 + \frac{1}{4}|b|^2$. Our main result on bifurcations of periodic/anti-periodic solutions is summarized below.

Theorem 1 *Let Assumption 1 be satisfied. Fix $n \in \mathbb{N}$, such that $w_{2n} \neq 0$. Let $\omega^2 = \frac{n^2}{4} + \epsilon\Omega$, where $\Omega \in \mathbb{R}$. The nonlinear elliptic problem (1.3) with $N = 1$ has a non-trivial solution $U(x)$ in the form*

(1.10) with $\mathbf{U} \in l_s^2(\mathbb{Z}')$ *for any* $s > \frac{1}{2}$ *and sufficiently small* ϵ *if and only if there exists a non-trivial solution for* $(a,b) \in \mathbb{C}^2$ *of the bifurcation equations*

$$\left\{ \begin{array}{l} \Omega a + w_{2n} b - \sigma(|a|^2 + 2|b|^2)a = \epsilon A_\epsilon(a,b), \\ \Omega b + w_{-2n} a - \sigma(2|a|^2 + |b|^2)b = \epsilon B_\epsilon(a,b), \end{array} \right. \tag{1.12}$$

where $A_\epsilon(a,b)$ *and* $B_\epsilon(a,b)$ *are analytic functions of* ϵ *near* $\epsilon = 0$ *satisfying the bounds*

$$\forall |\epsilon| < \epsilon_0,\ \forall |a| + |b| < \delta: \quad |A_\epsilon(a,b)| \leq C_A(|a|+|b|), \qquad |B_\epsilon(a,b)| \leq C_B(|a|+|b|), \tag{1.13}$$

for sufficiently small $\epsilon_0 > 0$*, fixed* $\delta > 0$*, and some constants* $C_A, C_B > 0$ *which are independent of* ϵ *and depend on* δ*. Moreover,* $A_\epsilon(a,b) = \bar{B}_\epsilon(b,a)$*,* $\forall (a,b) \in \mathbb{C}^2$ *and*

$$\forall |\epsilon| < \epsilon_0: \quad \left\| U(x) - \sqrt{\epsilon}\left(a e^{\frac{inx}{2}} + b e^{\frac{-inx}{2}}\right) \right\|_{C_b^0(\mathbb{R})} \leq C\epsilon^{3/2}. \tag{1.14}$$

for some ϵ*-independent constant* $C > 0$*.*

Corollary 1 *The coupled system (1.12) admits a symmetry reduction* $a = \bar{b}$*, where the value of* $a \in \mathbb{C}$ *satisfies the scalar equation*

$$\Omega a + w_{2n}\bar{a} - 3\sigma|a|^2 a = \epsilon A_\epsilon(a, \bar{a}). \tag{1.15}$$

Under the reduction, the solution $U(x)$ *is real-valued.*

The results of Theorem 1 and Corollary 1 justify the use of the y-independent coupled-mode system (1.6) for bifurcations of periodic/anti-periodic solutions of the nonlinear elliptic problem (1.3) with $N = 1$. In particular, the only non-zero solutions of the scalar equation (1.15) occur for either $a \in \mathbb{R}$ or $a \in i\mathbb{R}$, when the scalar equation (1.15) is reduced to the extended nonlinear dispersion relation

$$\Omega \pm w_{2n} - 3\sigma|a|^2 = \epsilon A_\pm(|a|), \tag{1.16}$$

where $A_\pm(|a|) = \frac{1}{a}A_\epsilon(a,\bar{a})$ is a bounded, real-valued error term for sufficiently small ϵ and finite value of $|a| \in \mathbb{R}$. Note that the values of $A_\pm$ are real-valued due to the gauge invariance of the coupled-mode system (1.12) inherited from the gauge invariance of the elliptic problem (1.3).

The newly formed gap (1.8) of the continuous spectrum of $L = -\partial_x^2 + \epsilon W(x)$ corresponds to the interval $|\Omega| < |w_{2n}|$. For instance, let $\sigma = +1$ and $w_{2n} > 0$, then the localized solution of the coupled-mode system (1.6) can be written in the exact form [6, 22]

$$a(y) = \bar{b}(y) = \frac{\sqrt{2}}{\sqrt{3}} \frac{\sqrt{w_{2n}^2 - \Omega^2}}{\sqrt{w_{2n} - \Omega}\cosh(\kappa y) + i\sqrt{w_{2n} + \Omega}\sinh(\kappa y)}, \tag{1.17}$$

where $\kappa = \frac{1}{n}\sqrt{w_{2n}^2 - \Omega^2}$. The exact localized solution can easily be found for $\sigma = -1$ and $w_{2n} < 0$. The trivial parameters of translations of solutions in y and $\arg(a)$ are set to zero in the explicit solution (1.17), such that the functions $a(y)$ and $b(y)$ satisfy the constraints $a(y) = \bar{a}(-y) = \bar{b}(y)$.

Definition 1 *The gap soliton of the coupled-mode system (1.6) is said to be a* reversible homoclinic orbit *if it decays to zero at infinity and satisfies the constraints* $a(y) = \bar{a}(-y) = \bar{b}(y)$*.*

We justify the persistence of the gap soliton (1.17) of the coupled-mode system (1.6) in the nonlinear elliptic problem (1.3) by working with the Fourier transform of $U(x)$

$$U(x) = \frac{\sqrt{\epsilon}}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{U}(k) e^{ikx} dk, \qquad \hat{U}(k) = \frac{1}{\sqrt{2\pi\epsilon}} \int_{\mathbb{R}} U(x) e^{-ikx} dx, \tag{1.18}$$

where again the factor $\sqrt{\epsilon}$ is introduced for convenience. We develop the method of Lyapunov–Schmidt reductions in the continuous weighted space $L^1_q(\mathbb{R})$ equipped with the norm

$$\hat{U} \in L^1_q(\mathbb{R}) : \quad \|\hat{U}\|_{L^1_q(\mathbb{R})} = \int_{\mathbb{R}} \left(1+k^2\right)^{q/2} |\hat{U}(k)| dk < \infty. \tag{1.19}$$

By the Riemann–Lebesgue Lemma, if $\hat{U} \in L^1_q(\mathbb{R})$ for some $q \geq 0$, then the n-th derivative of the function $U(x)$ is bounded and continuous for $0 \leq n \leq [q]$ and it decays to zero at infinity, i.e., $U \in C^n_b(\mathbb{R})$ and $\lim_{|x|\to\infty} U^{(n)}(x) = 0$. Indeed, for any $G(x) = U^{(n)}(x)$ and $\hat{G} \in L^1(\mathbb{R})$, it follows that

$$\|G(x)\|_{C^0_b(\mathbb{R})} \leq C \|\hat{G}(k)\|_{L^1(\mathbb{R})}, \tag{1.20}$$

for some $C > 0$. In addition, the Schwartz space is dense in $L^1(\mathbb{R})$ such that there is a sequence $\{\hat{G}_j(k)\}_{j\in\mathbb{N}}$ in the Schwartz space which converges to $\hat{G}(k)$ in L^1-norm, and therefore, there exists a sequence $\{G_j(x)\}_{j\in\mathbb{N}}$ which converges to $G(x)$ in $C^0_b(\mathbb{R})$-norm, such that $\lim_{|x|\to\infty} G(x) = 0$.

Related to the coupled-mode system for (a,b) in variable y, we shall also use the Fourier transform for $(\hat{a},\hat{b})$ in variable p, where $y = \epsilon x$ and $p = \frac{k}{\epsilon}$. We note that the norm $L^1(\mathbb{R})$ is invariant as follows:

$$\hat{A}(k) = \frac{1}{\epsilon} \hat{a}\left(\frac{k}{\epsilon}\right) : \qquad \|\hat{A}\|_{L^1(\mathbb{R})} = \|\hat{a}\|_{L^1(\mathbb{R})}. \tag{1.21}$$

This invariance explains the choice of the space $L^1_q(\mathbb{R})$ in our analysis (see also [21]). Our main result on the existence of gap soliton solutions is summarized below.

Theorem 2 *Let Assumption 1 be satisfied. Fix $n \in \mathbb{N}$, such that $w_{2n} \neq 0$. Let $\omega = \frac{n^2}{4} + \epsilon\Omega$, such that $|\Omega| < |w_{2n}|$. Let $a(y) = \bar{b}(y)$ be a reversible homoclinic orbit of the coupled-mode system (1.6) in Definition 1. Then, the nonlinear elliptic problem (1.3) with $N = 1$ has a non-trivial solution $U(x)$ in the form (1.18) with $\hat{U} \in L^1_q(\mathbb{R})$ for any $q \geq 0$ and sufficiently small ϵ such that*

$$\forall |\epsilon| < \epsilon_0 : \quad \|U(x) - \sqrt{\epsilon}\left[a(\epsilon x) e^{\frac{inx}{2}} + \bar{a}(\epsilon x) e^{-\frac{inx}{2}}\right]\|_{C^0_b(\mathbb{R})} \leq C\epsilon^{5/6}, \tag{1.22}$$

for some sufficient small ϵ_0 and ϵ-independent constant $C > 0$. Moreover, the solution $U(x)$ is real-valued, continuous on $x \in \mathbb{R}$, and $\lim_{|x|\to\infty} U(x) = 0$.

The results of Theorem 2 give more precise information about gap solitons of the elliptic problem (1.3) with $N = 1$ compared to the general result in Theorem 1.1 of [19], since the leading-order approximation of $U(x)$ is given by the exponentially decaying solutions (1.17) of the coupled-mode system (1.6). On the other hand, we do not prove in Theorem 2 that $U(x)$ decays exponentially as $|x| \to \infty$.

Rigorous justification of the approximation (1.5) and the time-dependent extensions of the coupled-mode system (1.6) were developed in [12] for the system of cubic Maxwell equations and in [21] for the Klein–Fock equation with quadratic nonlinearity. A bound on the error terms was found in the Sobolev space $H^1(\mathbb{R})$ in [12] and in the space of bounded continuous functions $C_b^0(\mathbb{R})$ in [21]. The bound is valid on a finite interval of the time evolution, which depends on ϵ. The error is not controlled on the entire time interval $t \in \mathbb{R}$ and, in particular, the formalism cannot be used for a proof of persistence of the leading-order approximation (1.5) and (1.17) for the stationary solutions of the nonlinear elliptic problem (1.3). The results of our Theorem 2 are more precise than Theorem 1 of [12] and Theorem 2.1 of [21] in this sense, since the error bound of the leading-order approximation is controlled independently of $t \in \mathbb{R}$. In a similar context, the justification of the nonlinear Schrödinger equation for the nonlinear Klein–Gordon equation with spatially periodic coefficients is reported in [4].

The method of Lyapunov–Schmidt reductions for periodic solutions was used in [8, 11]. The work [8] deals with a two-dimensional lattice equation for the nonlinear wave equation when eigenvalues of the relevant linearized operator accumulate near the origin. In this case, the Nash–Moser Theorem must be used for the infinite-dimensional part of the Lyapunov–Schmidt decomposition. In our case, the linearized operator for the one-dimensional lattice equation has eigenvalues bounded away of the origin and the Implicit Function Theorem can be applied without additional complications. This application of the technique is similar to the one in [11], which deals with the periodic wave solutions in the system of coupled discrete lattice equations. Other applications of the method for periodic wave solutions in equations of fluid dynamics can be found in [5, 7].

Persistence of modulated pulse solutions was considered in [13, 14] in the context of the nonlinear Klein–Gordon equations. (Earlier results on the same topics can be found in [3, 9].) Methods of spatial dynamics were applied to a relevant PDE problem for modulated pulse solutions, the linearization of which possessed infinitely many eigenvalues on the imaginary axis. The local center-stable manifold was constructed for the nonlinear Klein–Gordon equations after normal-form transformations and the pulse solutions were proved to be localized along a finite spatial scale, while small oscillatory tails occur generally beyond this spatial scale. In contrast to these works, we will not reformulate the ODE problem as an extended PDE problem and avoid the construction of the local center-stable manifold. This simplification is only possible if the variables of the time-dependent problems (1.1) and (1.2) can be separated and modulated pulse solutions are described by the reduction to the elliptic problem (1.3). We note that the basic equations of electrodynamics, such as the real-valued Maxwell equation (of the Klein–Gordon type), would not support the separation of variables and the modulated pulse solutions do not generally exist in the real-valued Maxwell equation [3].

The article is structured as follows. The proof of Theorem 1 is given in Section 2, where the technique of Lyapunov–Schmidt reductions in $l_s^2(\mathbb{Z}')$ is developed for bifurcations of periodic/anti-periodic solutions. The proof of Theorem 2 is given in Section 3, where the technique of Lyapunov–Schmidt reductions in $L_q^1(\mathbb{R})$ is extended for persistence of decaying solutions. Section 4 discusses applications of similar methods for the justification of multi-dimensional multi-component coupled-mode systems with $N \geq 2$.

2 Lyapunov–Schmidt reductions for periodic/anti-periodic solutions

Let the potential $W(x)$ and the solution $U(x)$ to the elliptic problem (1.3) with $N = 1$ be expanded in the Fourier series (1.4) and (1.10) respectively. By using the Fourier series, we convert the elliptic

problem (1.3) with $N=1$ in the space of bounded, continuous, periodic/anti-periodic solutions $U \in C_b^0(\mathbb{R})$ to a system of nonlinear difference equations in the discrete Sobolev weighted space $\mathbf{U} \in l_s^2(\mathbb{Z}')$ for some $s > \frac{1}{2}$. The nonlinear difference equations are written in the explicit form

$$\left(\omega^2 - \frac{m^2}{4}\right) U_m + \epsilon \sum_{m_1 \in \mathbb{Z}'} w_{m-m_1} U_{m_1} = \epsilon \sigma \sum_{m_1 \in \mathbb{Z}'} \sum_{m_2 \in \mathbb{Z}'} U_{m_1} \bar{U}_{-m_2} U_{m-m_1-m_2}, \qquad \forall m \in \mathbb{Z}', \tag{2.1}$$

which can be casted in the equivalent matrix-vector form

$$(\mathcal{L} + \epsilon \mathcal{W}) \, \mathbf{U} = \epsilon \sigma \mathbf{N}(\mathbf{U}, \bar{\mathbf{U}}, \mathbf{U}). \tag{2.2}$$

Here $\mathbf{U}$ is an element of the infinite-dimensional vector space $l_s^2(\mathbb{Z}')$ with the norm (1.9), elements of matrix operators $\mathcal{L}$ and $\mathcal{W}$ are given by

$$\mathcal{L}_{m,k} = \left(\omega^2 - \frac{m^2}{4}\right) \delta_{m,k}, \quad \mathcal{W}_{m,k} = w_{m-k}, \qquad \forall (m,k) \in \mathbb{Z}' \times \mathbb{Z}',$$

and $\mathbf{N}(\mathbf{U}, \bar{\mathbf{U}}, \mathbf{U}) = \mathbf{U} \star \mathcal{R}\bar{\mathbf{U}} \star \mathbf{U}$ consists of the convolution operator with the elements $(\mathbf{U} \star \mathbf{V})_m = \sum_{k \in \mathbb{Z}} U_k V_{m-k}$ and the inversion operator with the elements $(\mathcal{R}\mathbf{U})_m = U_{-m}$.

We shall verify that the nonlinear vector field associated with the difference equations (2.1) is closed in space $\mathbf{U} \in l_s^2(\mathbb{Z}')$ with $s > \frac{1}{2}$ (Lemma 1). Working in this space, we shall apply the Implicit Function Theorem in two cases $\omega \neq \mathbb{R} \backslash \{\frac{n}{2}\}_{n \in \mathbb{Z}}$ and $\omega = \omega_n = \frac{n}{2}$ for some $n \in \mathbb{N}$. The first case is non-resonant and the Implicit Function Theorem guarantees existence of the unique zero solution $\mathbf{U} = \mathbf{0}$ of system (2.2) near $\mathbf{U} = \mathbf{0}$ and $\epsilon = 0$ (Lemma 2). The second case corresponds to a bifurcation of non-zero periodic or anti-periodic solutions of system (2.2) and it is analyzed by using the Lyapunov–Schmidt decomposition. To prove Theorem 1, we will prove that there exists a unique smooth map from the components (U_n, U_{-n}) to the other components U_m, $\forall m \in \mathbb{Z}' \backslash \{n, -n\}$. Projections to the components (U_n, U_{-n}) yield the coupled-mode equations (1.12), while the bounds on the remainder terms (1.13) follow from the bounds on the vector $\mathbf{U}$ in space $l_s^2(\mathbb{Z}')$. Representation (1.14) and symmetry reductions of Corollary 1 follow from the technique of Lyapunov–Schmidt reductions.

Lemma 1 *Let $\mathbf{W} \in l_s^2(\mathbb{Z})$ for all $s > \frac{1}{2}$. The vector fields $\mathbf{W}\star$ and $\mathbf{N}$ map elements of $l_s^2(\mathbb{Z}')$ with $s > \frac{1}{2}$ to elements of $l_s^2(\mathbb{Z}')$.*

Proof. The space $l_s^2(\mathbb{Z})$ with $s > \frac{1}{2}$ forms a Banach algebra. Therefore, there exists a constant $0 < C(s) < \infty$ such that

$$\forall \mathbf{U}, \mathbf{V} \in l_s^2(\mathbb{Z}) : \quad \|\mathbf{U} \star \mathbf{V}\|_{l_s^2(\mathbb{Z})} \leq C(s) \|\mathbf{U}\|_{l_s^2(\mathbb{Z})} \|\mathbf{V}\|_{l_s^2(\mathbb{Z})}, \quad \forall s > \frac{1}{2}. \tag{2.3}$$

This property was proven in [8] and it is similar to the one in the continuous $H^s(\mathbb{R})$ spaces. The maps $\mathbf{W} \star \mathbf{U}$ and $\mathbf{U} \star \mathcal{R}\bar{\mathbf{U}} \star \mathbf{U}$ act on $\mathbf{U} \in l_s^2(\mathbb{Z}')$, where $\mathbb{Z}'$ is a set of either even or odd numbers. Both convolution operators transform a vector on $\mathbb{Z}'$ to a vector on $\mathbb{Z}'$. Therefore, $l_s^2(\mathbb{Z}')$ forms a linear subspace in the vector space $l^2(\mathbb{Z})$ with the same algebra property (2.3). As a result, both $\mathbf{W} \star \mathbf{U}$ and $\mathbf{N}(\mathbf{U}, \bar{\mathbf{U}}, \mathbf{U})$ map elements of $l_s^2(\mathbb{Z}')$ with $s > \frac{1}{2}$ to elements of $l_s^2(\mathbb{Z}')$. $\square$

Lemma 2 *Let $\mathbf{W} \in l_s^2(\mathbb{Z})$ with $s > \frac{1}{2}$ and $\omega \in \mathbb{R} \backslash \{\frac{n}{2}\}_{n \in \mathbb{Z}}$ for $n \in \mathbb{Z}$. The nonlinear lattice system (2.2) has a unique trivial solution $\mathbf{U} = \mathbf{0}$ in a local neighborhood of $\mathbf{U} = \mathbf{0}$ and $\epsilon = 0$.*

Proof. If $\omega \in \mathbb{R}\backslash\{\frac{n}{2}\}_{n\in\mathbb{Z}}$, the operator $\mathcal{L}$ in system (2.2) is invertible and

$$\|\mathcal{L}^{-1}\|_{l_s^2\mapsto l_s^2} \leq \frac{1}{\min\limits_{m\in\mathbb{Z}'} |\omega^2 - \frac{m^2}{4}|} = \rho_0 < \infty, \qquad \forall s \geq 0.$$

It follows from the estimate (1.11) that $\sum\limits_{k\in\mathbb{Z}'} |w_{m-k}| < \infty$ for any $m \in \mathbb{Z}'$ and $s > \frac{1}{2}$. Therefore, the matrix operator $\mathcal{W}$ is a relatively compact perturbation to $\mathcal{L}$. By the perturbation theory [15], there exists an ϵ-independent constant $0 < \rho < \infty$ such that

$$\|\left(\mathcal{L} + \epsilon\mathcal{W}\right)^{-1}\|_{l_s^2\mapsto l_s^2} \leq \rho, \quad \forall s \geq 0,$$

for sufficiently small ϵ. By Lemma 1, the vector field on the right-hand side of system (2.2) is closed in $l_s^2(\mathbb{Z}')$ for $s > \frac{1}{2}$. Moreover, it is analytic with respect to $\mathbf{U} \in l_s^2(\mathbb{Z}')$ and $\epsilon \in \mathbb{R}$. The zero solution $\mathbf{U} = \mathbf{0}$ satisfies the nonlinear lattice system (2.2) for any $\epsilon \in \mathbb{R}$. The Fréchet derivative of system (2.2) at $\mathbf{U} = \mathbf{0}$ (which is just the operator $\mathcal{L} + \epsilon W$) has a continuous bounded inverse for sufficiently small ϵ. By the Implicit Function Theorem, the zero solution $\mathbf{U} = \mathbf{0}$ is unique in a local neighborhood of $\mathbf{U} = \mathbf{0}$ and $\epsilon = 0$. □

Proof of Theorem 1. If $\omega = \frac{n}{2}$ for some $n \in \mathbb{N}$, the operator $\mathcal{L}$ is singular with a two-dimensional kernel

$$\mathrm{Ker}(\mathcal{L}) = \mathrm{Span}\left(\mathbf{e}_n, \mathbf{e}_{-n}\right) \subset l_s^2(\mathbb{Z}'),$$

where $\mathbf{e}_n$ is a unit vector in $l_s^2(\mathbb{Z}')$. The straightforward decomposition of $l_s^2(\mathbb{Z}') = \mathrm{Ker}(\mathcal{L}) \oplus \mathrm{Ker}(\mathcal{L})^{\perp}$ is nothing but the representation

$$\mathbf{U} = a\mathbf{e}_n + b\mathbf{e}_{-n} + \mathbf{g}, \tag{2.4}$$

where

$$\mathbf{g} \in \mathrm{Ker}(\mathcal{L})^{\perp} = \{\mathbf{g} \in l_s^2(\mathbb{Z}') : \quad g_n = g_{-n} = 0\}. \tag{2.5}$$

Let $\mathcal{P}$ be the projection operator from $l_s^2(\mathbb{Z}')$ to $\mathrm{Ker}(\mathcal{L})^{\perp}$ at $\omega^2 = \frac{n^2}{4}$. It is obvious that $\mathcal{PLP}$ is a non-singular operator at $\omega^2 = \frac{n^2}{4}$. By using the same argument as in Lemma 2, we obtain that there exists an ϵ-independent constant $0 < \rho < \infty$, such that

$$\|\left(\mathcal{P}\left(\mathcal{L} + \epsilon\mathcal{W}\right)\mathcal{P}\right)^{-1}\|_{l_s^2\mapsto l_s^2} \leq \rho, \quad \forall s \geq 0 \tag{2.6}$$

for sufficiently small ϵ. The inhomogeneous problem for $\mathbf{g}$ is written in the explicit form

$$\begin{aligned} m \neq \pm n : \qquad & \left(\frac{n^2 - m^2}{4} + \epsilon\Omega\right) g_m + \epsilon \sum_{k\in\mathbb{Z}'\backslash\{n,-n\}} w_{m-k} g_k \\ & - \epsilon\sigma \sum_{m_1\in\mathbb{Z}'} \sum_{m_2\in\mathbb{Z}'} U_{m_1} \bar{U}_{-m_2} U_{m-m_1-m_2} = -\epsilon\left(a w_{m-n} + b w_{m+n}\right), \end{aligned} \tag{2.7}$$

where $U_m = g_m + a\delta_{m,n} + b\delta_{m,-n}$. By Lemma 1, the vector field of system (2.7) is closed in $l_s^2(\mathbb{Z}')$ for $s > \frac{1}{2}$ and any $(a,b) \in \mathbb{C}^2$. Moreover, it is analytic with respect to $\mathbf{g} \in l_s^2(\mathbb{Z}')$ for all $(a,b) \in \mathbb{C}^2$ and $\epsilon \in \mathbb{R}$. By the bound (2.6) and the Implicit Function Theorem there exists a unique trivial solution $\mathbf{g} = \mathbf{0}$ of system (2.7) for $(a,b) = (0,0)$ and any $\epsilon \in \mathbb{R}$. It is also obvious that the zero solution exists for any $(a,b) \in \mathbb{C}^2$ and $\epsilon = 0$. For all $(a,b) \neq 0$, the Fréchet derivative of system (2.7) at $\mathbf{g} = \mathbf{0}$ is different from the matrix operator $\mathcal{P}(\mathcal{L} + \epsilon W)\mathcal{P}$ by the additional terms

$$-\epsilon\sigma\left[(|a|^2 + |b|^2)\delta_{m,k} + a\bar{b}\delta_{m,k+2n} + \bar{a}b\delta_{m,k-2n}\right], \qquad \forall (m,k) \in \mathbb{Z}' \times \mathbb{Z}'.$$

For sufficiently small ϵ and finite $(a,b) \in \mathbb{C}^2$, these terms change slightly the bound ρ in (2.6), such that the Fréchet derivative operator of system (2.7) at $\mathbf{g} = \mathbf{0}$ has a continuous bounded inverse for $|\epsilon| < \epsilon_0$, where ϵ_0 is sufficiently small. By the Implicit Function Theorem, there exists a unique map $\mathbf{G}_\epsilon : \mathbb{C}^2 \mapsto \mathrm{Ker}(\mathcal{L})^\perp \subset l_s^2(\mathbb{Z}')$, which is analytic in ϵ with the properties $\mathbf{G}_\epsilon(0,0) = \mathbf{0}$ and $\mathbf{G}_0(a,b) = \mathbf{0}$. Therefore, the map $\mathbf{G}_\epsilon$ admits the Taylor series expansion in ϵ. The first term of the Taylor series is

$$\mathbf{G}_\epsilon(a,b) = \epsilon \left[a\mathbf{g}_a + b\mathbf{g}_b + a^2\bar{b}\mathbf{g}_c + \bar{a}b^2\mathbf{g}_d \right] + \mathrm{O}(\epsilon^2),$$

where non-zero components of vectors $\mathbf{g}_{a,b,c,d}$ in the constrained space (2.5) are

$$(\mathbf{g}_a)_m = \frac{4w_{m-n}}{m^2-n^2}, \quad (\mathbf{g}_b)_m = \frac{4w_{m+n}}{m^2-n^2}, \quad (\mathbf{g}_c)_m = \frac{4\sigma\delta_{m,3n}}{m^2-n^2}, \quad (\mathbf{g}_d)_m = \frac{4\sigma\delta_{m,-3n}}{m^2-n^2}.$$

Let $|a| + |b| < \delta$ and δ is fixed independently of ϵ. Due to the analyticity of $\mathbf{G}_\epsilon$ in ϵ, there exists an ϵ-independent constant $C > 0$ such that

$$\forall |\epsilon| < \epsilon_0 : \quad \|\mathbf{G}_\epsilon(a,b)\|_{l_s^2(\mathbb{Z}')} \leq \epsilon C \left(|a| + |b| \right). \tag{2.8}$$

The projection equations to the two-dimensional kernel of $\mathcal{L}$ is found from system (2.1) at $m = \pm n$ in the explicit form

$$\begin{array}{c} (\Omega + w_0)a + w_{2n}b - \sigma \sum_{m_1 \in \mathbb{Z}'} \sum_{m_2 \in \mathbb{Z}'} U_{m_1}\bar{U}_{-m_2}U_{n-m_1-m_2} = -\sum_{k \in \mathbb{Z}' \backslash \{n,-n\}} w_{n-k}g_k, \\ (\Omega + w_0)b + w_{-2n}a - \sigma \sum_{m_1 \in \mathbb{Z}'} \sum_{m_2 \in \mathbb{Z}'} U_{m_1}\bar{U}_{-m_2}U_{-n-m_1-m_2} = -\sum_{k \in \mathbb{Z}' \backslash \{n,-n\}} w_{-n-k}g_k \end{array}, \tag{2.9}$$

where $U_m = a\delta_{m,n} + b\delta_{m,-n} + g_m$ and the map $\mathbf{g} = \mathbf{G}_\epsilon(a,b)$ is constructed above. At $\epsilon = 0$, we obtain that $\mathbf{g} = \mathbf{0}$ and

$$\begin{aligned} \sum_{m_1 \in \mathbb{Z}'} \sum_{m_2 \in \mathbb{Z}'} U_{m_1}\bar{U}_{-m_2}U_{n-m_1-m_2} &= \left(|a|^2 + 2|b|^2 \right) a, \\ \sum_{m_1 \in \mathbb{Z}'} \sum_{m_2 \in \mathbb{Z}'} U_{m_1}\bar{U}_{-m_2}U_{-n-m_1-m_2} &= \left(2|a|^2 + |b|^2 \right) b. \end{aligned}$$

This explicit computation recovers the left-hand side of system (1.12). The right-hand side is estimated from the bound (2.8) on the map $\mathbf{G}_\epsilon(a,b)$ in $l_s^2(\mathbb{Z}')$ with $s > \frac{1}{2}$ to yield the bound (1.13).

To prove the last assertion of Theorem 1, we shall prove that the map $\mathbf{G}_\epsilon(a,b)$ has the symmetry $(\mathbf{G}_\epsilon)_m(a,b) = (\bar{\mathbf{G}}_\epsilon)_{-m}(b,a)$. Indeed, since $W(x)$ is real-valued, its Fourier coefficients satisfy the constraint $w_{-2m} = \bar{w}_{2m}$, $\forall m \in \mathbb{Z}$. The systems of equations (2.7) and (2.9) are symmetric with respect to the transformation $(a,b,g_m) \mapsto (b,a,\bar{g}_{-m})$. By uniqueness of solutions of system (2.7) in a local neighborhood of $\epsilon = 0$, we obtain that $(\mathbf{G}_\epsilon)_m(a,b) = (\bar{\mathbf{G}}_\epsilon)_{-m}(b,a)$, $\forall m \in \mathbb{Z}'$. Then, it follows directly from system (2.9) that $A_\epsilon(a,b) = \bar{B}_\epsilon(b,a)$. □

Proof of Corollary 1. Due to the property $A_\epsilon(a,b) = \bar{B}_\epsilon(b,a)$, system (1.12) has the symmetry reduction $b = \bar{a}$, which results in the scalar equation (1.15). The vector $\mathbf{U}$ is given by the decomposition (2.4) with $b = \bar{a}$ and $(\mathbf{G}_\epsilon)_m(a,\bar{a}) = (\bar{\mathbf{G}}_\epsilon)_{-m}(\bar{a},a)$. Therefore, the solution $U(x)$ recovered from the Fourier series (1.10) is real-valued. □

3 Lyapunov–Schmidt reductions for gap solitons

Let the solution $U(x)$ to the elliptic problem (1.3) with $N=1$ be represented by the Fourier transform (1.18), while the potential $W(x)$ is given by the Fourier series (1.4). The elliptic problem (1.3) with $N=1$ is converted to the integral advance-delay equation for the Fourier transform $\hat{U}(k)$:

$$\left(\omega^2 - k^2\right)\hat{U}(k) + \epsilon \sum_{m\in\mathbb{Z}} w_{2m}\hat{U}(k-m) = \epsilon\sigma \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{U}(k_1)\hat{\bar{U}}(k_2)\hat{U}(k-k_1+k_2)dk_1dk_2 \qquad \forall k\in\mathbb{R}. \quad (3.1)$$

Working in the Fourier space $\hat{U}\in L^1_q(\mathbb{R})$, where the vector field of the integral advance-delay equation (3.1) is closed (Lemma 3), we decompose the solution $\hat{U}(k)$ into three parts

$$\hat{U}(k) = \hat{U}_+(k)\chi_{\mathbb{R}'_+}(k) + \hat{U}_-(k)\chi_{\mathbb{R}'_-}(k) + \hat{U}_0(k)\chi_{\mathbb{R}'_0}(k), \quad (3.2)$$

where $\chi_{[a,b]}(k)$ is a function of compact support (it is 1 on $k\in[a,b]$ and 0 on $k\in\mathbb{R}\backslash[a,b]$) and the intervals $\mathbb{R}'_+$, $\mathbb{R}'_-$ and $\mathbb{R}'_0$ are

$$\mathbb{R}'_+ = \left[\omega_n - \epsilon^{2/3}, \omega_n + \epsilon^{2/3}\right], \quad \mathbb{R}'_- = \left[-\omega_n - \epsilon^{2/3}, -\omega_n + \epsilon^{2/3}\right], \quad \mathbb{R}'_0 = \mathbb{R}\backslash(\mathbb{R}'_+\cup\mathbb{R}'_-). \quad (3.3)$$

Here $\omega_n = \frac{n}{2}$ is a bifurcation value of ω and the components $\hat{U}_\pm(k)$ represent the largest part of the solution $\hat{U}(k)$ near the resonant values $k=\pm\omega_n$, which is approximated by the solution of the coupled-mode system (1.6) in coordinates $y=\epsilon x$ in physical space and $p=\frac{k}{\epsilon}$ in Fourier space. The intervals surrounding the resonant values $k=\pm\omega_n$ have small length $2\epsilon^{2/3}$, where both the constant $c=2$ and the scaling factor $r=\frac{2}{3}$ are fixed for convenience. In fact, we could generalize all proofs for any constant $c>0$ and any scaling factor $\frac{1}{2}<r<1$.

In order to prove Theorem 2, we shall apply the method of Lyapunov–Schmidt reductions in space $L^1_q(\mathbb{R})$ with $q\geq 0$. First, we prove the existence of a unique smooth map from $(\hat{U}_+(k),\hat{U}_-(k))$ to $\hat{U}_0(k)$ (Lemma 4). The solutions for the components $(\hat{U}_+(k),\hat{U}_-(k))$ are then approximated by the suitable (exponentially decaying) solutions $(\hat{a}(p),\hat{b}(p))$ of the coupled-mode system (1.6) rewritten in Fourier space (Lemma 5). The approximation yields the desired bound (1.22) for $\hat{U}\in L^1_q(\mathbb{R})$ with $q\geq 0$. The reduction to the real-valued solutions $U(x)$ becomes obvious from the decomposition of the Lyapunov–Schmidt reduction method. Continuity and decay conditions on $U(x)$ follow by the Riemann–Lebesgue Lemma.

Lemma 3 *Let $\mathbf{W}\in l^2_{s+q}(\mathbb{Z})$ for all $s>\frac{1}{2}$ and $q\geq 0$. The vector field of the integral equation (3.1) maps elements of $L^1_q(\mathbb{R})$ with $q\geq 0$ to elements of $L^1_q(\mathbb{R})$.*

Proof. The convolution sum in the integral equation (3.1) is closed due to the bound

$$\forall\hat{U}\in L^1_q(\mathbb{R}),\ \forall\mathbf{W}\in l^2_{s+q}(\mathbb{Z}): \quad \left\|\sum_{m\in\mathbb{Z}} w_{2m}\hat{U}(k-m)\right\|_{L^1_q(\mathbb{R})} \leq \|\hat{U}\|_{L^1_q(\mathbb{R})}\|\mathbf{W}\|_{l^1_q(\mathbb{Z})} \leq \|\hat{U}\|_{L^1_q(\mathbb{R})}\|\mathbf{W}\|_{l^2_{s+q}(\mathbb{Z})}$$

for any $q\geq 0$ and $s>\frac{1}{2}$, where the inequality (1.11) has been used. The convolution integral is closed due to the bound

$$\forall\hat{U},\hat{V}\in L^1_q(\mathbb{R}): \quad \left\|\int_{\mathbb{R}}\hat{U}(k_1)\hat{V}(k-k_1)dk_1\right\|_{L^1_q(\mathbb{R})} \leq C\|\hat{U}\|_{L^1_q(\mathbb{R})}\|\hat{V}\|_{L^1_q(\mathbb{R})}$$

for any $q \geq 0$ and some $C > 0$, which occur in the inequality

$$1 + (k_1 + k_2)^2 \leq C(1 + k_1^2)(1 + k_2^2)$$

for all $(k_1, k_2) \in \mathbb{R}^2$. $\square$

Lemma 4 *Let Assumption 1 be satisfied and $\omega = \frac{n^2}{4} + \epsilon\Omega$, where $n \in \mathbb{N}$ and $\Omega \in \mathbb{R}$. There exists a unique map $\hat{U}_\epsilon : L^1_q(\mathbb{R}'_+) \times L^1_q(\mathbb{R}'_-) \mapsto L^1_q(\mathbb{R}'_0)$ for all $q \geq 0$, such that $\hat{U}_0(k) = \hat{U}_\epsilon(\hat{U}_+, \hat{U}_-)$ and*

$$\forall |\epsilon| < \epsilon_0 : \quad \|\hat{U}_0(k)\|_{L^1_q(\mathbb{R}'_0)} \leq \epsilon^{1/3} C \left(\|\hat{U}_+\|_{L^1_q(\mathbb{R}'_+)} + \|\hat{U}_-\|_{L^1_q(\mathbb{R}'_-)} \right), \tag{3.4}$$

where ϵ_0 is sufficiently small and the constant $C > 0$ is independent of ϵ and depends on δ in the bound $\|\hat{U}_+\|_{L^1_q(\mathbb{R}'_+)} + \|\hat{U}_-\|_{L^1_q(\mathbb{R}'_-)} < \delta$ for a fixed ϵ-independent $\delta > 0$.

Proof. We project the integral advance-delay equation (3.1) onto the interval $k \in \mathbb{R}'_0$:

$$\left(\frac{n^2}{4} + \epsilon\Omega - k^2 \right) \hat{U}_0(k) + \epsilon \sum_{m \in \mathbb{Z}} w_{2m} \chi_{\mathbb{R}'_0}(k) \hat{U}(k - m) = \epsilon\sigma \chi_{\mathbb{R}'_0}(k) \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{U}(k_1) \hat{\bar{U}}(k_2) \hat{U}(k - k_1 + k_2) dk_1 dk_2,$$

where $\hat{U}(k)$ is decomposed by the representation (3.2). Since

$$\min_{k \in \mathbb{R}'_0} \left| \frac{n^2}{4} - k^2 \right| \geq C_n \epsilon^{2/3},$$

for some $C_n > 0$, the linearized integral equation at $\epsilon = 0$ is invertible such that

$$\left\| \left(\frac{n^2}{4} - k^2 \right)^{-1} \right\|_{L^1_q(\mathbb{R}'_0) \mapsto L^1_q(\mathbb{R}'_0)} \leq \frac{1}{C_n \epsilon^{2/3}}. \tag{3.5}$$

The linearized integral equation on $\hat{U}_0(k)$ for $\epsilon \neq 0$ is given on $k \in \mathbb{R}'_0$ by

$$\begin{aligned}
\left(\frac{n^2}{4} + \epsilon\Omega - k^2 \right) \hat{U}_0(k) + \epsilon \sum_{m \in \mathbb{Z}} w_{2m} \chi_{\mathbb{R}'_0}(k) \hat{U}_0(k - m) \\
-\epsilon\sigma \chi_{\mathbb{R}'_0}(k) \int_{\mathbb{R}'_+} \int_{\mathbb{R}'_-} \left[\hat{U}_+(k_1) \hat{\bar{U}}_-(k_2) + \hat{\bar{U}}_+(k_1) \hat{U}_-(k_2) \right] \hat{U}_0(k + k_1 + k_2) dk_1 dk_2 \\
-\epsilon\sigma \chi_{\mathbb{R}'_0}(k) \int_{\mathbb{R}'_+} \int_{\mathbb{R}'_+} \hat{U}_+(k_1) \hat{\bar{U}}_+(k_2) \hat{U}_0(k - k_1 + k_2) dk_1 dk_2 \\
-\epsilon\sigma \chi_{\mathbb{R}'_0}(k) \int_{\mathbb{R}'_-} \int_{\mathbb{R}'_-} \hat{U}_-(k_1) \hat{\bar{U}}_-(k_2) \hat{U}_0(k - k_1 + k_2) dk_1 dk_2.
\end{aligned}$$

Recall that $\epsilon^{2/3} \gg \epsilon$ for sufficiently small ϵ. If $\mathbf{W} \in l^2_{s+q}(\mathbb{Z})$ and $\hat{U}_\pm \in L^1_q(\mathbb{R}'_\pm)$ for all $s > \frac{1}{2}$ and $q \geq 0$, the convolution sums and integrals are closed by Lemma 3. Fix $\|\hat{U}_+\|_{L^1_q(\mathbb{R}'_+)} + \|\hat{U}_-\|_{L^1_q(\mathbb{R}'_-)} < \delta$ for some ϵ-independent δ. Then, the linearized operator is continuously invertible for all $|\epsilon| < \epsilon_0$. The integral equation is analytic in ϵ and admits a unique trivial solution $\hat{U}_0(k) = 0$ if $\hat{U}_\pm(k) = 0$ on $k \in \mathbb{R}'_\pm$ or if $\epsilon = 0$. By the Implicit Function Theorem, there exists a unique map $\hat{U}_\epsilon : L^1_q(\mathbb{R}'_+) \times L^1_q(\mathbb{R}'_-) \mapsto L^1_q(\mathbb{R}'_0)$ for $|\epsilon| < \epsilon_0$. The map is analytic in ϵ near $\epsilon = 0$ with the properties $\hat{U}_0(\hat{U}_+, \hat{U}_-) = 0$ and $\hat{U}_\epsilon(0, 0) = 0$. Due to the analyticity of the map $\hat{U}_\epsilon(\hat{U}_+, \hat{U}_-)$ in ϵ and the bound (3.5) on the inverse operator, the solution $\hat{U}_0(k) = \hat{U}_\epsilon(\hat{U}_+, \hat{U}_-)$ satisfies the desired bound (3.4). $\square$

Lemma 5 *Let Assumption 1 be satisfied. Fix $n \in \mathbb{N}$, such that $w_{2n} \neq 0$. Let $\omega = \frac{n^2}{4} + \epsilon\Omega$, such that $|\Omega| < |w_{2n}|$. Let $a(y) = \bar{b}(y)$ be a reversible homoclinic orbit of the coupled-mode system (1.6) in Definition 1. Then, there exists a solution of the integral equation (3.1), such that $\hat{U}_0(k) = U_\epsilon(\hat{U}_+, \hat{U}_-)$ is given by Lemma 4 and*

$$\forall |\epsilon| < \epsilon_0 : \left\| \hat{U}_+(k) - \frac{1}{\epsilon}\hat{a}\left(\frac{k-\omega_n}{\epsilon}\right) \right\|_{L^1_q(\mathbb{R}'_+)} \leq C_a \epsilon^{1/3}, \ \left\| \hat{U}_-(k) - \frac{1}{\epsilon}\hat{b}\left(\frac{k+\omega_n}{\epsilon}\right) \right\|_{L^1_q(\mathbb{R}'_-)} \leq C_b \epsilon^{1/3}, \quad (3.6)$$

for sufficiently small $\epsilon_0 > 0$ and ϵ-independent constants $C_a, C_b > 0$.

Proof. Let us use the scaling invariance (1.21) and map the intervals $\mathbb{R}'_\pm$ for $\hat{U}_\pm(k)$ to the normalized interval $\mathbb{R}_0 = \left[-\epsilon^{-1/3}, \epsilon^{-1/3}\right]$ for

$$\hat{a}(p) = \epsilon \hat{U}_+\left(\frac{k-\omega_n}{\epsilon}\right), \qquad \hat{b}(p) = \epsilon \hat{U}_-\left(\frac{k+\omega_n}{\epsilon}\right). \quad (3.7)$$

The new functions $\hat{a}(p)$ and $\hat{b}(p)$ have a compact support on $p \in \mathbb{R}_0$, while the norms $\|\hat{a}\|_{L^1_q(\mathbb{R}_0)}$ and $\|\hat{b}\|_{L^1_q(\mathbb{R}_0)}$ are equivalent to the norms $\|\hat{U}_+\|_{L^1_q(\mathbb{R}'_+)}$ and $\|\hat{U}_-\|_{L^1_q(\mathbb{R}'_-)}$. Using the bound (3.4), we project the integral equation (3.1) to the system of two integral equations on $p \in \mathbb{R}_0$:

$$\begin{aligned}
&(\Omega + w_0 - np)\,\hat{a}(p) + w_{2n}\hat{b}(p) - \sigma \int_{\mathbb{R}_0}\int_{\mathbb{R}_0} \left[\hat{a}(p_1)\hat{\bar{a}}(p_2) + \hat{b}(p_1)\hat{\bar{b}}(p_2)\right] \hat{a}(p-p_1+p_2)dp_1 dp_2 \\
&\quad -\sigma \int_{\mathbb{R}_0}\int_{\mathbb{R}_0} \hat{a}(p_1)\hat{\bar{b}}(p_2)\hat{b}(p-p_1+p_2)dp_1dp_2 = \epsilon p^2 \hat{a}(p) + \epsilon^{1/3}\hat{A}_\epsilon(\hat{a},\hat{b},\hat{U}_\epsilon(\hat{a},\hat{b})), \quad (3.8)\\
&(\Omega + w_0 + np)\,\hat{b}(p) + w_{-2n}\hat{a}(p) - \sigma \int_{\mathbb{R}_0}\int_{\mathbb{R}_0} \left[\hat{a}(p_1)\hat{\bar{a}}(p_2) + \hat{b}(p_1)\hat{\bar{b}}(p_2)\right] \hat{b}(p-p_1+p_2)dp_1 dp_2 \\
&\quad -\sigma \int_{\mathbb{R}_0}\int_{\mathbb{R}_0} \hat{b}(p_1)\hat{\bar{a}}(p_2)\hat{a}(p-p_1+p_2)dp_1dp_2 = \epsilon p^2 \hat{b}(p) + \epsilon^{1/3}\hat{B}_\epsilon(\hat{a},\hat{b},\hat{U}_\epsilon(\hat{a},\hat{b})), \quad (3.9)
\end{aligned}$$

where A_ϵ and B_ϵ are computed from $(\hat{a},\hat{b})$ and the map $\hat{U}_\epsilon(\hat{a},\hat{b})$ of Lemma 4. When the right-hand side of system (3.8)–(3.9) is truncated and the integration is extended to $p \in \mathbb{R}$, system (3.8)–(3.9) becomes the coupled-mode system (1.6) rewritten after the Fourier transform in y. If $w_{2n} \neq 0$ and $|\Omega| < |w_{2n}|$, the coupled-mode system (1.6) has a reversible homoclinic orbit $a(y) = \bar{b}(y)$. The Fourier transform $\hat{a}(p)$ decays exponentially as $|p| \to \infty$, such that the integrals of the system (3.8)–(3.9) on $p \in \mathbb{R}\backslash\mathbb{R}_0$ are exponentially small in ϵ. Therefore, they can be moved to the right-hand side of the system. In addition, the remainder terms A_ϵ and B_ϵ are analytic in ϵ and controlled by the bound (3.4) on $(\hat{a}(p),\hat{b}(p))$ in the space $L^1_q(\mathbb{R}_0)$ for all $q \geq 0$. The linear terms $p^2\hat{a}(p)$ and $p^2\hat{b}(p)$ are also controlled by the bound

$$\epsilon\|p^2\hat{a}(p)\|_{L^1_q(\mathbb{R}_0)} \leq \epsilon^{1/3}\|\hat{a}(p)\|_{L^1_q(\mathbb{R}_0)}, \qquad \epsilon\|p^2\hat{b}(p)\|_{L^1_q(\mathbb{R}_0)} \leq \epsilon^{1/3}\|\hat{b}(p)\|_{L^1_q(\mathbb{R}_0)}. \quad (3.10)$$

Therefore, system (3.8)–(3.9) is a perturbed coupled-mode system (1.6) in Fourier space with the truncation error of the order $\mathrm{O}(\epsilon^{1/3})$ measured in space $L^1_q(\mathbb{R}_0)$. (Note that the system (3.8)–(3.9) is not closed on $L^1_q(\mathbb{R})$ as the terms $p^2\hat{a}(p)$ and $p^2\hat{b}(p)$ represent a singular second-order perturbation of the first-order coupled-mode system.)

To prove the persistence of decaying solutions of the coupled-mode system (1.6), we note that the linearized differential operator associated to the coupled-mode system (1.6) in the physical space is given by a self-adjoint system of 4-by-4 component Dirac operators:

$$\begin{bmatrix} in\partial_y + W_0 & -\sigma a^2 & w_{2n} - 2\sigma a\bar{b} & -2\sigma ab \\ -\sigma\bar{a}^2 & -in\partial_y + W_0 & -2\sigma\bar{a}\bar{b} & \bar{w}_{2n} - 2\sigma a\bar{b} \\ \bar{w}_{2n} - 2\sigma\bar{a}b & -2\sigma ab & -in\partial_y + W_0 & -\sigma b^2 \\ -2\sigma\bar{a}\bar{b} & w_{2n} - 2\sigma a\bar{b} & -\sigma\bar{b}^2 & in\partial_y + W_0 \end{bmatrix}, \tag{3.11}$$

where $W_0 = \Omega - 2\sigma(|a|^2 + |b|^2)$. By Theorem 4.1 and Corollary 4.2 in [6], the linearized operator (3.11) is block-diagonalized into two uncoupled 2-by-2 Dirac operators, each has a one-dimensional kernel. The two-dimensional kernel of the linearized operator (3.11) is related to the translational symmetries in y and $\arg(a)$ with the eigenvectors $[a'(y), b'(y), \bar{a}'(y), \bar{b}'(y)]^T$ and $[ia(y), ib(y), -i\bar{a}(y), -i\bar{b}(y)]^T$. The zero eigenvalue of the linearized operator (3.11) is bounded away from the continuous spectrum and other eigenvalues on the real axis [6]. The extended coupled-mode system given by the system (3.8)–(3.9) after the Fourier transform is only solvable if the right-hand-side lies in the range of the linearized operator (3.11).

The nonlinear elliptic problem (1.3) with real-valued symmetric potential $W(x) = W(-x) = \bar{W}(x)$ has two symmetries: the gauge invariance $U(x) \to e^{i\alpha}U(x)$ for all $\alpha \in \mathbb{R}$ and the reversibility $U(x) \to U(-x)$. The new system obtained after the Fourier transform (1.18) and the decomposition (3.2) inherits both symmetries, such that the extended coupled-mode system is formulated in a constrained subspace orthogonal to the kernel of the linearized operator (3.11). As a result, the linearized operator is continuously invertible in the constrained subspace of space $L^1_q(\mathbb{R}_0) \times L^1_q(\mathbb{R}_0)$ for all $q \geq 0$. Truncation of the integral terms introduces a small error in the remainder terms but does not change the symmetries of the extended coupled-mode system and does not alter the invertibility of the linearized operator. By the Implicit Function Theorem, there exists a unique solution of system (3.8)–(3.9) for $\hat{a}(p)$ and $\hat{b}(p)$ on $p \in \mathbb{R}_0$, which is close to the reversible homoclinic orbit of the coupled-mode system (1.6) in L^1_q-norm. $\square$

Remark 1 If $w_{2n} > 0$ and $\Omega = w_{2n}$, the exact solution (1.17) describes an algebraically decaying reversible homoclinic orbit of the coupled-mode system (1.6). Since the continuous spectrum of the linearized operator (3.11) touches the zero eigenvalue in this case, persistence of algebraically decaying reversible homoclinic orbits can not be proved in Lemma 5.

Remark 2 The symmetry condition on the potential $W(-x) = W(x)$ in Assumption 1 is important for the proof of persistence of homoclinic orbits in the nonlinear elliptic problem (1.3) since it ensures that the set of homoclinic orbits of the nonlinear problem (1.3) includes the symmetric (reversible) homoclinic orbits $U(-x) = U(x)$. Lemma 5 can not be proved if the right-hand-side of the extended coupled-mode system is not in the range of the linearized operator (3.11), which may occur when a homoclinic orbit of the coupled-mode system (1.6) is positioned arbitrarily with respect to the general potential function $W(x)$. Non-persistence of such homoclinic orbits is usually beyond all orders in the asymptotic expansion in powers of ϵ and it is typical that the gap solitons persist at two particular points on the period of the potential $W(x)$ (see [20] for details). In our paper, we avoid the beyond-all-orders problem by imposing a symmetry condition on $W(x)$ which is sufficient for existence of a reversible homoclinic orbit which is centered at the point $x = 0$.

Proof of Theorem 2. When the solution $U(x)$ is represented by the Fourier transform $\hat{U}(k)$, both scaling transformations of Lemma 4 and 5 are incorporated into the solution, and the bounds (3.4) and (3.6) are used, we obtain the bound

$$\forall |\epsilon| < \epsilon_0 : \quad \left\| \hat{U}(k) - \frac{1}{\epsilon} \hat{a} \left(\frac{k - n/2}{\epsilon} \right) - \frac{1}{\epsilon} \hat{b} \left(\frac{k + n/2}{\epsilon} \right) \right\|_{L^1_q(\mathbb{R})} \leq C \epsilon^{1/3}, \tag{3.12}$$

which implies the desired bound (1.22) in original physical space. It remains to prove that the solution $U(x)$ is real-valued. This property follows from the symmetry of the map $U_0(k) = \hat{U}_\epsilon(\hat{U}_+, \hat{U}_-)$ constructed in Lemma 4 with respect to the interchange of $\hat{U}_+(k - \omega_n)$ and $\bar{\hat{U}}_-(k + \omega_n)$ and the complex conjugation. As a result, system (3.8)–(3.9) has the symmetry reduction $\hat{a}(p) = \bar{\hat{b}}(p)$, which is satisfied by the solution of the truncated system. When the partition (3.2) is substituted into the Fourier transform (1.18) with the symmetry $\hat{U}_+(k - \omega_n) = \bar{\hat{U}}_-(k + \omega_n)$, the resulting solution $U(x)$ is proved to be real-valued. □

4 Lyapunov–Schmidt reductions in multi-dimensional potentials

Let us consider the elliptic problem (1.3) in the space of two dimensions ($N = 2$). Let the potential $W(x)$ be periodic in both variables with the same normalized period, such that

$$W(x_1 + 2\pi, x_2) = W(x_1, x_2 + 2\pi) = W(x_1, x_2), \qquad \forall (x_1, x_2) \in \mathbb{R}^2. \tag{4.1}$$

We shall justify the use of multi-component coupled-mode systems for the analysis of bifurcations of two-dimensional periodic/anti-periodic solutions of the elliptic problem (1.3). We use the Fourier series for the potential $W(x)$ and the solution $U(x)$:

$$W(x) = \sum_{m \in \mathbb{Z}^2} w_{2m} e^{i m \cdot x}, \qquad U(x) = \sqrt{\epsilon} \sum_{m \in \mathbb{Z}_1' \times \mathbb{Z}_2'} U_m e^{\frac{i}{2} m \cdot x}, \tag{4.2}$$

where $m \cdot x = m_1 x_1 + m_2 x_2$ and the sets $\mathbb{Z}_1'$ and $\mathbb{Z}_2'$ are even or odd if the solution $U(x)$ is periodic or anti-periodic in the corresponding variable x_1 and x_2. The elliptic problem (1.3) with $N = 2$ transforms to a system of nonlinear difference equations, which is similar to system (2.1):

$$\left(\omega^2 - \frac{|m|^2}{4} \right) U_m + \epsilon \sum_{m_1 \in \mathbb{Z}_1' \times \mathbb{Z}_2'} w_{m - m_1} U_{m_1} = \epsilon \sigma \sum_{m_1 \in \mathbb{Z}_1' \times \mathbb{Z}_2'} \sum_{m_2 \in \mathbb{Z}_1' \times \mathbb{Z}_2'} U_{m_1} \bar{U}_{-m_2} U_{m - m_1 - m_2}, \tag{4.3}$$

for all $m \in \mathbb{Z}_1' \times \mathbb{Z}_2'$. The nonlinear lattice system (4.3) is closed in the space $l^2_s(\mathbb{Z}_1' \times \mathbb{Z}_2')$ with $s > 1$ thanks to the Banach algebra property:

$$\forall \mathbf{U}, \mathbf{V} \in l^2_s(\mathbb{Z}^2) : \quad \|\mathbf{U} \star \mathbf{V}\|_{l^2_s(\mathbb{Z}^2)} \leq C(s) \|\mathbf{U}\|_{l^2_s(\mathbb{Z}^2)} \|\mathbf{V}\|_{l^2_s(\mathbb{Z}^2)}, \quad \forall s > 1, \tag{4.4}$$

for some $C(s) > 0$. Under the same constraint $s > 1$, the double Fourier series (4.2) converges absolutely and uniformly in $C^0_b(\mathbb{R}^2)$.

The system (4.3) takes the same abstract form (2.2), where $\mathbf{U}$ is an element of the vector space $l^2_s(\mathbb{Z}_1' \times \mathbb{Z}_2')$ with $s > 1$. An extension of Lemma 2 tells us that no non-trivial solution $\mathbf{U}$ of the system (4.3) exists in a local neighborhood of $\mathbf{U} = \mathbf{0}$ and $\epsilon = 0$ unless $\omega = \omega_n = \frac{|n|}{2}$, where $n = (n_1, n_2) \in \mathbb{Z}^2$

and $|n| = \sqrt{n_1^2 + n_2^2}$. Bifurcations of non-trivial solutions occur only in the resonant case $\omega = \omega_n$ and the number of bifurcation equations (leading to the coupled-mode system) is defined by the dimension of the resonant set S_n in

$$S_n = \left\{ m \in \mathbb{Z}_1' \times \mathbb{Z}_2' : \quad |m|^2 = |n|^2 \right\}. \tag{4.5}$$

Here again the set $\mathbb{Z}_1'$ is even/odd if n_1 is even/odd and so is the set $\mathbb{Z}_2'$ with respect to n_2.

Lemma 6 *The set S_n admits the following properties:*

(i) $0 < \mathrm{Dim}(S_n) < \infty$.

(ii) If $\mathbf{n} = \mathbf{0}$, *the zero solution* $\mathbf{m} = \mathbf{0}$ *is unique.*

(iii) If $\mathbf{n} = (n_1, 0)$, *then* $\mathrm{Dim}(S_n) \geq 2$ *if* n_1 *is odd and* $\mathrm{Dim}(S_n) \geq 4$ *if* n_1 *is even.*

(iv) If $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$, *then* $\mathrm{Dim}(S_n) \geq 4$ *if* $n_1 - n_2$ *is odd and* $\mathrm{Dim}(S_n) \geq 8$ *if* $n_1 - n_2$ *is even and non-zero.*

Proof. (i) follows from the bound $|m|^2 < \infty$ on the space of integers and from the existence of the solution $m = n$. (ii) is obvious from $|m|^2 = 0$. (iii) follows from the existence of particular solutions $(\pm n_1, 0)$ and $(0, \pm n_1)$ of $|m|^2 = n_1^2$ (if n_1 is odd, the solutions $(0, \pm n_1)$ do not belong to the space $\mathbb{Z}_2'$ of even numbers). (iv) follows from the existence of particular solutions $(\pm n_1, \pm n_2)$ and $(\pm n_2, \pm n_1)$ of $|m|^2 = n_1^2 + n_2^2$ (if $n_1 - n_2$ is odd, the solutions $(\pm n_2, \pm n_1)$ do not belong to the space $\mathbb{Z}_1' \times \mathbb{Z}_2'$ of the opposite parities and if $n_1 = n_2$, the solutions $(\pm n_2, \pm n_1)$ are not different from $(\pm n_1, \pm n_2)$). $\square$

Proposition 1 *Let* $\mathbf{W} \in l_s^2(\mathbb{Z}^2)$ *for all* $s > 1$ *and* $\omega^2 = \frac{|n|^2}{4} + \epsilon\Omega$ *for some* $n \in \mathbb{N}^2$ *and* $\Omega \in \mathbb{R}$. *Let the set* S_n *be defined by (4.5) with* $d_S = \mathrm{Dim}(S_n)$. *The nonlinear lattice system (4.3) has a non-trivial solution* $\mathbf{U} \in l_s^2(\mathbb{Z}_1' \times \mathbb{Z}_2')$ *for all* $s > 1$ *and sufficiently small* ϵ *if and only if there exists a non-trivial solution for* $\mathbf{a} \in \mathbb{C}^{d_S}$ *of the bifurcation equations*

$$\Omega a_{j_m} + \sum_{m_1 \in S_n} w_{m-m_1} a_{j_{m_1}} - \sigma \sum_{m_1 \in S_n} \sum_{-m_2 \in S_n'} a_{j_{m_1}} \bar{a}_{j_{-m_2}} a_{j_{m-m_1-m_2}} = \epsilon A_{j_m,\epsilon}(\mathbf{a}), \quad \forall m \in S_n, \tag{4.6}$$

where j_m *is an index of* m *in the set* S_n, *the set* S_n' *is a subset of* S_n, *such that* $m - m_1 - m_2 \in S_n$, $\mathbf{A}_\epsilon(\mathbf{a}) \in \mathbb{C}^{d_S}$ *depends analytically on* ϵ *near* $\epsilon = 0$. *Moreover, there exists constant* $C > 0$ *which is independent of* $\epsilon_0 > 0$ *and depends on* $\delta > 0$ *such that*

$$\forall |\epsilon| < \epsilon_0, \;\; \forall \|\mathbf{a}\|_{l^1(\mathbb{C}^{d_S})} < \delta : \quad \|\mathbf{A}_\epsilon(\mathbf{a})\|_{l^1(\mathbb{C}^{d_S})} \leq C \|\mathbf{a}\|_{l^1(\mathbb{C}^{d_S})}, \tag{4.7}$$

where ϵ_0 *is sufficiently small and* δ *is fixed independently of* ϵ_0.

Proof. The proof repeats the proof of Theorem 1 due to the fact that $|m|^2 - |n|^2$ are bounded away from zero for $m \in \mathbb{Z}_1' \times \mathbb{Z}_2' \backslash S_n$. The Lyapunov–Schmidt reductions are performed after the decomposition $\mathbf{U} = \sum_{m \in S_n} a_{j_m} \mathbf{e}_m + \mathbf{g}$, where $\mathbf{g} \in \mathrm{Ker}(\mathcal{L})^\perp$. $\square$

Example 1 The resonant value $\omega_{(0,0)} = 0$ corresponds to the single-mode bifurcation, like in the one-dimensional problem ($N = 1$). The next resonant value $\omega_{(1,0)} = \omega_{(0,1)} = \frac{1}{2}$ corresponds to the two-mode

bifurcation, which has the same coupled-mode equations as in the one-dimensional problem ($N = 1$) due to the separation of the periodic Fourier series in one variable and the anti-periodic Fourier series in the other variable. Finally, the next resonant value $\omega_{(1,1)} = \frac{1}{\sqrt{2}}$ gives the first example of the non-trivial four-component coupled-mode equations in the space of two dimensions ($N = 2$). The coupled-mode equations (4.6) can be rewritten explicitly for the components (a_1, a_2, a_3, a_4) which corresponds to the Fourier modes for the resonant set $S_{(1,1)} = \{(1, 1); (-1, -1); (1, -1); (-1, 1)\}$ at the selected order:

$$\begin{aligned}
(\Omega + w_{0,0})a_1 + w_{2,2}a_2 + w_{0,2}a_3 + w_{2,0}a_4 &= \sigma\left((|a_1|^2 + 2|a_2|^2 + 2|a_3|^2 + 2|a_4|^2)a_1 + 2\bar{a}_2a_3a_4\right), \\
(\Omega + w_{0,0})a_2 + w_{-2,-2}a_1 + w_{-2,0}a_3 + w_{0,-2}a_4 &= \sigma\left((2|a_1|^2 + |a_2|^2 + 2|a_3|^2 + 2|a_4|^2)a_2 + 2\bar{a}_1a_3a_4\right), \\
(\Omega + w_{0,0})a_3 + w_{2,-2}a_4 + w_{0,-2}a_1 + w_{2,0}a_2 &= \sigma\left((2|a_1|^2 + 2|a_2|^2 + |a_3|^2 + 2|a_4|^2)a_3 + 2\bar{a}_4a_1a_2\right), \\
(\Omega + w_{0,0})a_4 + w_{-2,2}a_3 + w_{-2,0}a_1 + w_{0,2}a_2 &= \sigma\left((2|a_1|^2 + 2|a_2|^2 + 2|a_3|^2 + |a_4|^2)a_4 + 2\bar{a}_3a_1a_2\right).
\end{aligned}$$

This system (with the derivative terms in $y_1 = \epsilon x_1$ and $y_2 = \epsilon x_2$) was derived in [1] by using asymptotic multi-scale expansions. Higher-order resonances for ω_n with larger values of $n \in \mathbb{N}^2$ may involve more than four components in the coupled-mode equations (4.6), and the count of d_S versus n is not available in general.

Remark 3 Additional resonances were considered in [1], which correspond to an oblique propagation of the resonant Fourier modes, e.g. $e^{\frac{i}{2}px_1}$ and $e^{\frac{i}{2}((p+2m_1)x_1+2m_2x_2)}$ with $p = -(m_1^2+m_2^2)/m_1 \notin \mathbb{Z}$. These resonances can be incorporated in the present analysis by using the transformation $U(x) = e^{\frac{i}{2}px_1}\tilde{U}(x)$.

Remark 4 Existence of two-dimensional ($N = 2$) gap soliton solutions can not be proved with the approach of Section 3 for small values of ϵ when ω is close to ω_n. Indeed, if the solution $\hat{U}(k)$ is split into a finite number of parts compactly supported near the points of resonances $(k_1, k_2) = \frac{1}{2}(n_1, n_2)$ and the remainder part $\hat{U}_0(\mathbb{R})$, then the operator $|k|^2 - \omega_n^2$ with $\omega_n^2 = \frac{|n|^2}{4}$ is not invertible in a neighborhood of the circle of the radius $|k| = \frac{|n|}{2}$. Since only finitely many parts of the circle are excluded from the compact support of $\hat{U}_0(k)$, the Implicit Function Theorem can not be used to prove existence of the map from the finitely many resonance parts of the solution $\hat{U}(k)$ to the remainder part $\hat{U}_0(k)$. This obstacle has a principal nature as it is related to a generic non-existence of gap solitons in the systems without spectral gaps. Indeed, the operator $L = -\nabla^2 - \epsilon W(x)$ has no gaps for sufficiently small ϵ in the space of two dimensions ($N = 2$) [17].

Remark 5 The conclusions of Proposition 1 and Remark 4 can be extended to $N \geq 3$.

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References

[1] D. Agueev and D. Pelinovsky, ”Modeling of wave resonances in low-contrast photonic crystals”, SIAM J. Appl. Math. **65**, 1101–1129 (2005)

[2] S. Alama and Y.Y. Li, "Existence of solutions for semilinear elliptic equations with indefinite linear part", J. Diff. Eqs. **96**, 89–115 (1992)

[3] G. Alfimov and V.V. Konotop, "On the existence of gap solitons", Physica D **146**, 307–327 (2000)

[4] K. Busch, G. Schneider, L. Tkeshelashvili, and H. Uecker, "Justification of the nonlinear Schrödinger equation in spatially periodic media", Z. Angew. Math. Phys. **57**, 905-939 (2006)

[5] H. Chen, "Existence of periodic travelling-wave solutions of nonlinear, dispersive wave equations", Nonlinearity **17**, 2041–2056 (2004)

[6] M. Chugunova and D. Pelinovsky, "Block-diagonalization of the symmetric first-order coupled-mode system", SIAM J. Appl. Dyn. Syst. **5**, 66-83 (2006)

[7] W. Craig and D.P. Nicholls, "Traveling two and three dimensional capillary gravity water waves", SIAM J. Math. Anal. **32**, 323–359 (2000)

[8] W. Craig and C.E. Wayne, "Newton's method and periodic solutions of nonlinear wave equations", Commun. Pure Appl. Math. **46**, 1409–1498 (1993)

[9] V.M. Eleonsky, N.E. Kulagin, N.S. Novozhilova, and V.P. Silin, "Solutions of wave equations that are self-localized in space and time-periodic", Selecta Math. Soviet. **7**, 1–14 (1988).

[10] H.P. Heinz, T. Kupper, and C.A. Stuart, "Existence and bifurcation of solutions for nonlinear perturbations of the periodic Schrödinger equation", J. Diff. Eqs. **100**, 341–354 (1992)

[11] A. Georgieva, T. Kriecherbauer, and S. Venakides, "Wave propagation and resonance in a one-dimensional nonlinear discrete periodic medium", SIAM J. Appl. Math. **60**, 272-294 (1999)

[12] R.H. Goodman, M.I. Weinstein, and P.J. Holmes, "Nonlinear propagation of light in one-dimensional periodic structures", J. Nonlinear. Science **11**, 123–168 (2001)

[13] M.D. Groves and G. Schneider, "Modulating pulse solutions for a class of nonlinear wave equations", Commun. Math. Phys. **219**, 489–522 (2001)

[14] M.D. Groves and G. Schneider, "Modulating pulse solutions for quasilinear wave equations", J. Diff. Eqs. **219**, 221–258 (2005)

[15] T. Kato, "*Perturbation theory for linear operators*", Springer-Verlag (1976)

[16] *"Emergent Nonlinear Phenomena in Bose–Einstein Condensates"*, Eds. P.G. Kevrekidis, D.J. Franzeskakis, and R. Carretero–Gonzalez (Springer–Verlag, New York, 2007)

[17] P. Kuchment, "The mathematics of photonic crystals", in *"Mathematical Modeling in Optical Sciences"* (SIAM, Philadelphia, 1999)

[18] T. Kupper and C.A. Stuart, "Necessary and sufficient conditions for gap-bifurcation", Nonlinear Anal. **18**, 893–903 (1992)

[19] A. Pankov, "Periodic nonlinear Schrdinger equation with application to photonic crystals", Milan J. Math. **73**, 259–287 (2005)

[20] D.E. Pelinovsky, A.A. Sukhorukov, and Yu.S. Kivshar, Bifurcations and stability of gap solitons in periodic potentials, Phys. Rev. E **70**, 036618 (2004)

[21] G. Schneider and H. Uecker, "Nonlinear coupled mode dynamics in hyperbolic and parabolic periodically structured spatially extended systems", Asymp. Anal. **28**, 163–180 (2001)

[22] C.M. de Sterke and J.E. Sipe, "Gap solitons", Progress in Optics, **33**, 203 (1994)